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Generalized multivalued vector variational-like inequalities

Syed Shakaib Irfan · Rais Ahmad

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Abstract In this article, we consider a generalized multivalued vector variational-like inequality and obtain some existence results. The last result is proved by using the concept of escaping sequences. Some special cases are also discussed.

Keywords Generalized multivalued vector variational-like inequality · Existence result · Escaping sequence · Closed graph · Affine mapping

Mathematics Subject Classification (2000) 49J40 · 47H19 · 47H10

1 Introduction and preliminaries

The vector variational inequality is a generalized form of a variational inequality, having applications in different areas of optimization, optimal control, operations research, economics equilibrium and free boundary value problems. It was introduced by [8] in finite dimensional Euclidean space in 1980. Since then, in a general setting [4], [5], [3] have derived an equivalance between the vector variational inequality and vector complementarity problem and proved the existence of solutions to the vector variational inequality.

In 1990, [6] introduced and studied vector variational inequality for multivalued mappings which they called generalized vector variational inequality. In 1993, [12] studied generalized vector variational inequalities in reflexive Banach spaces. In 2000, [9] studied the vector variational inequality and have shown its equivalence with vector equilibria.

S. S. Irfan (✉)
College of Engineering, Qassim University, P. O. Box 6677, Buraidah, Al-Qassim 51452, Saudi Arabia
e-mail: shakaib11@rediffmail.com

R. Ahmad
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
e-mail: raisain@lycos.com

The aim of this article is to derive two existence theorems for generalized multivalued vector variational-like inequalities.The last existence result is proved by using the concept of escaping sequences introduced in [2].

Let Y be Banach space with a convex cone P such that $int P \neq \phi$ and $P \neq Y$, where int denotes the interior. We use the following vector ordering:

(1) for any $x, y \in Y$, $y < x$ if and only if $y - x \in -\text{int}P$;
(2) for any $x, y \in Y$, $y \not< x$ if and only if $y - x \notin -\text{int}P$;
(3) for any two sets $A, B \subset Y$, $A < B$ if and only if $a < b$ for any $a \in A$ and any $b \in B$;
(4) for any two sets $A, B \subset Y$, $A \not< B$ if and only if $a \not< b$ for any $a \in A$ and any $b \in B$.

Let X be a nonempty subset of a Banach space E and Y a Banach space with a convex cone P such that $int P \neq \phi$ and $P \neq Y$. Let $T : K \to 2^{L(E,Y)}$ be a set-valued map, where $L(E, Y)$ is the space of all linear continuous mapping from E into Y, $\eta : X \times X \to Y$ and $g : X \to Y$ be the mappings.

We consider the following *generalized multivalued vector variational-like inequality* (GMVVLI):

$$\text{(GMVVLI)} \quad \begin{cases} \text{Find } x_0 \in X \text{ such that} \\ \langle s_0, \eta(x, x_0)\rangle + g(x) - g(x_0) \not< 0 \text{ for any } x \in X \text{ and } s_0 \in T(x_0). \end{cases}$$

where $\langle s, x\rangle$ is the evaluation of s at x.

If $\eta(x, x_0) = x - x_0$, then (GMVVLI) reduces to the following *generalized vector variational inequality* (GVVI), introduced and studied by [11].

$$\text{(GVVI)} \quad \begin{cases} \text{Find } x_0 \in X \text{ such that} \\ \langle s_0, x - x_0\rangle + g(x) - g(x_0) \not< 0 \text{ for any } x \in X \text{ and } s_0 \in T(x_0). \end{cases}$$

If $g \equiv 0$ and $\eta(x, x_0) = x - x_0$, then (GMVVLI) collapses to the following *vector variational inequality*:

$$\text{(VVI)} \quad \begin{cases} \text{Find } x_0 \in X \text{ such that} \\ \langle s_0, x - x_0\rangle \not< 0 \text{ for any } s_0 \in T(x_0) \end{cases}$$

(VVI) is the same as the vector variational inequality introduced in [10] which was considered in an H-Banach Space.

Now we give some definitions and KKM-Fan Theorem needed for the proof of the existence results.

Definition 1.1 [7]. Let X be a subset of a topological space E. Then a set-valued map $F : X \to 2^E$ is called the KKM map if for each finite subset $\{x_1, x_2, \dots x_n\}$ of X, $\text{Co}\{x_1, x_2, \dots x_n\} \subset \bigcup_{i=1}^n F(x_i)$, where $\text{Co}\{x_1, x_2, \dots x_n\}$ is the convex hull of $\{x_1, x_2, \dots x_n\}$.

Definition 1.2 [2]. Let E be a topological space and X be a subset of E, such that $X = \bigcup_{n=1}^\infty X_n$, where $\{X_n\}_{n=1}^\infty$ is an increasing sequence of nonempty compact sets in the sense that $X_n \subseteq X_{n+1}$ for all $n \in N$. A sequence $\{x_n\}_{n=1}^\infty$ in X is said to be escaping sequence X (relative to $\{X_n\}_{n=1}^\infty$), if for each $n = 1, 2, \dots$, there exist $m > 0$ such that $x_k \notin X_n$ for all $k \geq m$.

Theorem 1.1 (KKM-FAN) [7]. *Let X be a subset of a topological space E and $F : X \to 2^E$ a KKM map. If for each $x \in X$, $F(x)$ is closed and for at least one $x \in X$, $F(x)$ is compact, then $\bigcap_{x \in E} F(x) \neq \phi$.*

2 Existence results

In this section, we prove two existence results for (GMVVLI). The last results is proved by using the concept of escaping sequences.

Theorem 2.1 *Let X be a compact convex subset of Banach space E and Y a Banach space with convex cone P such that $\mathrm{int}P \neq \phi$ and $P \neq Y$. Assume that:*

(i) *$T : X \to 2^{L(E,Y)}$ is a lower semicontinuous mapping*;
(ii) *$g : X \to Y$ is a continuous mapping*;
(iii) *$\eta : X \times X \to Y$ is a continuous mapping such that $\eta(x, x) = 0$ for all $x \in X$*;
(iv) *the multivalued mapping $W : X \to 2^Y$ defined by $W(x) = Y/\{-\mathrm{int}P\}$, has a closed graph in $X \times Y$*;
(v) *for each $y \in X,\ B_y := \{x \in X :$ there exists $s \in T(y)$such that $\langle s, \eta(x, y)\rangle + g(x) - g(y) < 0\}$ is convex.*

Then the generalized multivalued vector variational-like inequality (GMVVLI) is solvable.

Proof Define a multivalued mapping $F : X \to 2^E$ by:

For any $x \in X$, $F(x) = \{y \in X : \langle s_0, \eta(x, y)\rangle + g(x) - g(y) \not< 0$ for any $s_0 \in T(y)\}$.

We first prove that F is a KKM map. Suppose, to the contrary, F is not a KKM-map. Then the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is not contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$.

Let y be an element in the convex hull of $\{x_1, x_2, \ldots, x_n\}$. Then $y = \sum_{i=1}^n \alpha_i x_i$ for some $\alpha_i \geq 0,\ i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \alpha_i = 1$ and y is not contained in $\bigcup_{i=1}^n F(x_i)$. Then, we have $\forall\, i \in \{1, 2, \ldots, n\}\ \exists\, s_0 \in T(y)$ such that

$$\langle s_0, \eta(x_i, y)\rangle + g(x_i) - g(y) < 0.$$

Since by assumption (v), B_y is convex, the convex hull of $\{x_1, x_2, \ldots x_n\}$ is contained in B_y.

We have

$$\langle s_0, \eta(\Sigma_{i=1}^n \alpha_i x_i, \Sigma_{i=1}^n \alpha_i x_i)\rangle + g(\Sigma_{i=1}^n \alpha_i x_i) - g(\Sigma_{i=1}^n \alpha_i y_i) \in -\mathrm{int}P.$$

Thus, $0 \in -\mathrm{int}P$, but this contradicts $P \neq Y$. Therefor F is a KKM map.

Next we prove that for any $x \in X,\ F(x)$ is closed. Indeed, let $\{y_n\}$ be a sequence in $F(x)$ converging to $y_* \in X$. By the lower semicontinuity of T, for any $s_* \in T(y_*)$, there exists $s_n \in T(y_n)$ for all n such that the sequence $\{s_n\}$ converging to $s_* \in L(E, Y)$. Since $y_n \in F(x)$ for all n, we have

$$\langle s_n, \eta(x, y_n)\rangle + g(x) - g(y_n) \not< 0$$

or

$$\langle s_n, \eta(x, y_n)\rangle + g(x) - g(y_n) \in W(y_n).$$

Since $\{s_n\}$ is bounded in $L(E, Y),\ \eta(.,.),\ \langle .,.\rangle$ and g are continuous. Also since W has a closed graph in $X \times Y$ and $s_n \to s_*, y_n \to y_*$, we have

$$\langle s_n, \eta(x, y_n)\rangle + g(x) - g(y_n) \longrightarrow \langle s_*, \eta(x, y_*)\rangle + g(x) - g(y_*) \in W(y_*).$$

Hence $\langle s_*, \eta(x, y_*)\rangle + g(x) - g(y_*) \not< 0$. Thus $y_* \in F(x)$ and $F(x)$ is closed.

Further, since X is a compact subset of E and $F(y_0) \subset X$ for each $y_0 \in X$. Hence $F(y_0)$ is compact. Therefore, the assumptions of Theorem 1.1 hold. By Theorem 1.1, $\bigcap_{x\in X} F(x) \neq \phi$ and hence there exists $x_0 \in X$ such that

$$\langle s_0, \eta(x, x_0)\rangle + g(x) - g(x_0) \nless 0 \quad \text{for any } x \in X \text{and any } s_0 \in T(x_0).$$

The assumption (v) in Theorem 2.1 is strong. We can remove assumption (v) in Theorem 2.1 with some additional assumptions on η, g and W, where η, g and W are defined in (ii), (iii) and (iv) of the Theorem 2.1, respectively. Thus, we have a Corollary as follows. □

Corollary 2.1 *Let X be a compact convex subset of Banach space E and Y a Banach space with convex cone P such that int$P \neq \phi$ and $P \neq Y$. Assume that*:

(i) *$T : X \to 2^{L(E,Y)}$ is a lower semicontinuous mapping*;
(ii) *$g : X \to Y$ is a continuous affine mapping*;
(iii) *$\eta : X \times X \to Y$ is a continuous affine mapping such that $\eta(x, x) = 0$ for all $x \in X$;*
(iv) *the multivalued mapping $W : X \to 2^Y$ defined by $W(x) = Y/\{-\text{int}P\}$, has a closed graph in $X \times Y$ and W is convex.*

Then the generalized multivalued vector variational-like inequality (*GMVVLI*) *is solvable.*

Proof It is sufficient to prove that for each $y \in X$, the set $B_y = \{x \in X : \langle s_0, \eta(x, y)\rangle + g(x) - g(y) < 0$ for any $s_0 \in T(y)\}$ is convex. To see this, let $x_1, x_2 \in B_y$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Then for each $s_0 \in T(y)$ we have

$$\langle s_0, \eta(x_1, y)\rangle + g(x_1) - g(y) \in -\text{intP} \tag{1}$$

and

$$\langle s_0, \eta(x_2, y)\rangle + g(x_2) - g(y) \in -\text{intP} \tag{2}$$

multiplying (1) by α and (2) by β and adding, we have

$$\alpha\langle s_0, \eta(x_1, y)\rangle + \alpha g(x_1) - \alpha g(y) + \beta\langle s_0, \eta(x_2, y)\rangle + \beta g(x_2) - \beta g(y) \notin \alpha W(y) + \beta W(y).$$

Since $\eta(.,.)$, g are affine and W is concave, we have

$$\langle s_0, \eta(\alpha x_1 + \beta x_2, y)\rangle + g(\alpha x_1 + \beta x_2) - g(y) \in W(y)$$

or

$$\langle s_0, \eta(\alpha x_1 + \beta x_2, y)\rangle + g(\alpha x_1 + \beta x_2) - g(y) < 0$$

and hence B_y is convex. □

Remark 2.1 (i) Theorem 2.1 generalizes and improves the corresponding results in [1,11].
(ii) Corrollary 2.1 is a generaliztion of corrollary 2.1 in [11].

Theorem 2.2 *Let X be a compact convex subset of Banach space E and Y a Banach space with convex cone P such that int$P \neq \phi$ and $P \neq Y$. Let $X = \bigcup_{n=1}^{\infty} X_n$ where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact sets in the sense that $X_n \subseteq X_{n+1}$ for all $n \in N$. Assume that*:

(i) *$T : X \to 2^{L(E,Y)}$ is a lower semicontinuous mapping*;
(ii) *$g : X \to Y$ is a continuous mapping*;
(iii) *$\eta : X \times X \to Y$ is a continuous mapping such that $\eta(x, x) = 0$ for all $x \in X$;*
(iv) *the multivalued mapping $W(x) = Y/\{\text{-int}P\}$ has a closed graph in $x \times Y$;*

(v) *for each* $y \in X$, $B_y := \{x \in X :$ *there exists* $\in T(y)$*such that* $\langle s, \eta(x, y)\rangle + g(x) - g(y) < 0\}$ *is convex.*

(vi) *for each sequence* $\{y_n\}_{n=1}^{\infty}$ *in* X *with* $y_n \in X_n$, $n \in N$ *which is escaping from* X *relative to* $\{X_n\}_{n=1}^{\infty}$ *there exists* $m \in N$ *and* $x_m \in X_m$ *such that*

$$\langle s_m, \eta(x_m, y_m)\rangle + g(x_m) - g(y_m) < 0.$$

Then there exists $y^* \in X$ *such that*

$$\langle s^*, \eta(x, y^*)\rangle + g(x) - g(y^*) \not< 0, \quad \textit{for any } s^* \in T(y^*).$$

Proof Since for each $n \in N$, X_n is compact and convex set in E, applying Theorem 2.1, we have for all $n \in N$, there exists $y_n \in X_n$ such that

$$\langle s_n, \eta(x, y_n)\rangle + g(x) - g(y_n) \not< 0, \quad \text{for all } s_n \in T(y_n) \tag{3}$$

suppose that the sequence $\{y_n\}_{n=1}^{\infty}$ be escaping sequence from X relative to $\{X_n\}_{n=1}^{\infty}$. By (vi) there exists $m \in N$ and $x_m \in X_n$ such that

$$\langle s_m, \eta(x_m, y_m)\rangle + g(x_m) - g(y_m) < 0,$$

which contradicts (3). Hence $\{y_n\}_{n=1}^{\infty}$ is not an escaping sequence from X relative to $\{X_n\}_{n=1}^{\infty}$. Therefore, there exists $r \in N$ and there is some subsequence $\{y_{j_n}\}$ of $\{y_n\}_{n=1}^{\infty}$ which must lie entirely in X_r. Since X_r is compact, there is a subsequence $\{y_{i_n}\}_{i_n \in \wedge}$ of $\{y_{j_n}\}$ in X_r and there exists $y^* \in X_r$ such that $y_{i_n} \to y^*$, where $i_n \to \infty$. Since $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence we have for all $x \in X$ there exists $i_0 \in \wedge$ with $i_0 > r$, such that $x \in X_{i_0}$ for all $i_n \in \wedge$ and $i_n > i_0$, we have $x \in X_{i_0} \subseteq X_{i_n}$ and $T(y_{i_n}) \subseteq T(X_r)$ such that

$$\langle s_{i_n}, \eta(x, y_{i_n})\rangle + g(x) - g(y_{i_n}) \not< 0, \ \text{for any } s_{i_n} \in T(y_{i_n}),$$

which implies that

$$\langle s_{i_n}, \eta(x, y_{i_n})\rangle + g(x) - g(y_{i_n}) \in W(y_{i_n}),$$

using the same argument as in Theorem 2.1, we have

$$\langle s^*, \eta(x, y^*)\rangle + g(x) - g(y^*) \not< 0.$$

The result follows. □

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